



Dual group actions on C^* -algebras and their description by Hilbert extensions

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Abstract

Given a C^* -algebra \mathcal{A} , a discrete abelian group \mathcal{X} and a homomorphism $\Theta: \mathcal{X} \rightarrow \text{Out } \mathcal{A}$, defining the dual action group $\Gamma \subset \text{aut } \mathcal{A}$, the paper contains results on existence and characterization of Hilbert extensions of $\{\mathcal{A}, \Gamma\}$, where the action is given by \mathcal{X} . They are stated at the (abstract) C^* -level and can therefore be considered as a refinement of the extension results given for von Neumann algebras for example by Jones [16] or Sutherland [20, 21]. A Hilbert extension exists iff there is a generalized 2-cocycle. These results generalize those in [10], which are formulated in the context of superselection theory, where it is assumed that the algebra \mathcal{A} has a trivial center, i.e. $\mathcal{Z} = \mathbb{C}\mathbb{1}$. In particular the well-known “outer characterization” of the second cohomology $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \alpha_{\mathcal{X}})$ can be reformulated: there is a bijection to the set of all \mathcal{A} -module isomorphism classes of Hilbert extensions. Finally, a Hilbert space representation (due to Sutherland [20, 21] in the von Neumann case) is mentioned. The C^* -norm of the Hilbert extension is expressed in terms of the norm of this representation and it is linked to the so-called regular representation appearing in superselection theory.

1 Introduction

In the Doplicher/Roberts theory (e.g. [12, 14]) it is a central assumption that the center of the C^* -algebra \mathcal{A} with which one starts the analysis is trivial, i.e. $\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$ (in a more categorical notation the assumption reads $(\iota, \iota) = \mathbb{C}\mathbb{1}$, where ι denotes the unit object of the strict monoidal C^* -category, cf. [13]). From a systematical point of view it is interesting to study the properties and structural modifications of this theory if one assumes the presence of a nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$. For example, if $(\mathcal{F}, \alpha_{\mathcal{G}})$ is a Hilbert C^* -system for a compact group \mathcal{G} and if the corresponding fixed point algebra \mathcal{A} has a nontrivial center that satisfies the relation $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, then the Galois correspondence does not hold anymore, i.e. we have the proper inclusion $\alpha_{\mathcal{G}} \subset \text{stab } \mathcal{A}$ in $\text{aut } \mathcal{F}$ (cf. [6, Section 7]). Recall, that in the trivial center situation it is a fundamental result of the theory that $\alpha_{\mathcal{G}} = \text{stab } \mathcal{A}$. As a further justification we can also mention that in other generalizations of the Doplicher/Roberts theory as well as in some applications in mathematical physics a nontrivial center plays, to a certain extent, a distinguished role [17, 24, 15].

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In the present paper we continue the analysis of the presence of a nontrivial center in the construction of an extension algebra \mathcal{F} (cf. [4, 5]). In particular, we study what we call dual group actions in the simple case where the group \mathcal{X} is discrete and abelian (cf. with [10] in the special case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$). This investigations will be done at the abstract C^* -level which is the context of the Doplicher/Roberts theory mentioned above (cf. also [3]). On the other hand the results can be considered as a refinement of the study of twisted group algebras (twisted crossed products) on the concrete von Neumann algebra level (see e.g. [9, 16, 20, 21]). For example, the decisive C^* -norm for the extension is defined intrinsically and the natural representation (discussed e.g. by Sutherland) is related to the so-called regular representation that appears in the superselection theory [2]. We hope that the present analysis will be useful to obtain a more general ‘inversion’ theorem, where endomorphisms of \mathcal{A} are involved. Indeed, the main theorems in Section 3 suggest that for a more general inversion theory in the nontrivial center situation the cohomological aspects may be essential.

The paper is structured in 5 sections: in the following section we will introduce the notion of a Hilbert C^* -system and study some properties of the group homomorphism $\Theta: \mathcal{X} \rightarrow \text{Out } \mathcal{A}$. Hilbert C^* -systems are the result of the extension procedure mentioned above. In Section 3 we begin the study of the inverse (extension) problem: in particular it contains the result that a Hilbert extension exists iff there is a generalized 2-cocycle (to be defined there), and that in this case the set of all Hilbert extensions can be described in terms of the set of center-valued 2-cocycles of $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \alpha_{\mathcal{X}})$ (cf. Theorems 3.4 and 3.8). In the next section we relate the previously obtained results to the special case of the Doplicher/Roberts frame, where $\mathcal{Z} = \mathbb{C}\mathbb{1}$. Finally, in Section 5 we give a representation of the Hilbert extension, which was already studied by Sutherland [20, 21] in the von Neumann case. In particular, we show that if there is a faithful state of \mathcal{A} , this representation coincides with the so-called regular representation that appears in superselection theory (cf. e.g. [2]) and the intrinsic C^* -norm turns out to be the operator norm of this representation.

2 Hilbert C^* -systems

A C^* -algebra \mathcal{F} together with a pointwise norm-continuous group homomorphism $\mathcal{G} \ni g \rightarrow \alpha_g \in \text{aut } \mathcal{F}$ of a locally compact group \mathcal{G} is called a C^* -system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$. Let $\mathcal{A} \subseteq \mathcal{F}$ be its fixed point algebra, i.e. $\mathcal{A} := \{A \in \mathcal{F} \mid \alpha_g A = A, g \in \mathcal{G}\}$. We denote by $\mathcal{A}^c := \mathcal{F} \cap \mathcal{A}' \subseteq \mathcal{F}$ the relative commutant of \mathcal{A} w.r.t. \mathcal{F} . As is well-known, $\alpha_g|_{\mathcal{A}^c}$ is an automorphism of \mathcal{A}^c , so $\{\mathcal{A}^c, \alpha_{\mathcal{G}}\}$ is also a C^* -system. We call it the *assigned* C^* -system. The center $\mathcal{Z}(\mathcal{A})$ is denoted by \mathcal{Z} .

In the following let \mathcal{G} be compact and abelian so that $\hat{\mathcal{G}} =: \mathcal{X}$ is abelian and discrete. The corresponding spectral projections w.r.t. $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ are denoted by Π_{χ} , $\chi \in \mathcal{X}$. Note that $\Pi_{\iota}\mathcal{F} = \mathcal{A}$, where ι is the unit element of \mathcal{X} .

2.1 Definition A C^* -system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$, \mathcal{G} compact abelian, is called a Hilbert C^* -system if $\text{spec } \alpha_{\mathcal{G}} = \mathcal{X}$ and if each spectral subspace $\Pi_{\chi}\mathcal{F}$ contains a unitary U_{χ} , i.e. $\mathcal{U}(\Pi_{\chi}\mathcal{F}) \neq \emptyset$.

If $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ is Hilbert, then $\beta_{\chi} := \text{ad } U_{\chi}|_{\mathcal{A}}$ is an automorphism of \mathcal{A} , i.e. $\beta_{\chi} \in \text{aut } \mathcal{A}$. We denote by π the canonical homomorphism of $\text{aut } \mathcal{A}$ onto $\text{Out } \mathcal{A} := \text{aut } \mathcal{A} / \text{int } \mathcal{A}$, where $\text{int } \mathcal{A}$ denotes the normal subgroup of all inner automorphisms of \mathcal{A} . Then

$$\mathcal{X} \ni \chi \rightarrow \Theta(\chi) := \pi(\beta_{\chi}) \in \text{Out } \mathcal{A} \quad (1)$$

is a group homomorphism of \mathcal{X} into $\text{Out } \mathcal{A}$, i.e. we have

2.2 Lemma To each Hilbert C^* -system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$, where \mathcal{G} is compact abelian, there is canonically assigned a group homomorphism $\Theta: \mathcal{X} \rightarrow \text{Out } \mathcal{A}$ given by (1).

Proof: Note that for $\chi_1, \chi_2 \in \mathcal{X}$ we have that $U_{\chi_1\chi_2}U_{\chi_2}^*U_{\chi_1}^* \in \mathcal{A}$ and this implies that $\beta_{\chi_1\chi_2} \circ \beta_{\chi_2}^{-1} \circ \beta_{\chi_1}^{-1} \in \text{int } \mathcal{A}$. ■

We mention next the characterization of those Hilbert C^* -systems where Θ is an isomorphism and of those where the classes $\Theta(\chi)$ are pairwise disjoint. Recall that $\alpha, \beta \in \text{aut } \mathcal{A}$ are called disjoint if

$$(\alpha, \beta) := \{X \in \mathcal{A} \mid X\alpha(A) = \beta(A)X \text{ for all } A \in \mathcal{A}\} = 0.$$

2.3 Proposition (i) Θ is a monomorphism iff no spectral subspace $\Pi_\chi \mathcal{A}^c, \chi \neq \iota$, of the assigned C^* -system contains a unitary.

(ii) The classes $\Theta(\chi)$ are pairwise disjoint iff $\mathcal{A}^c = \mathcal{Z}$, i.e. the relative commutant coincides with the center of \mathcal{A} .

Proof: For one of the directions of part (i) take a unitary $U_\chi \in \Pi_\chi(\mathcal{A}^c)$ with $\iota \neq \chi \in \mathcal{X}$, so that the corresponding $\beta_\chi = \text{id}$ and $\pi(\beta_\chi) = \text{int } \mathcal{A}$. Thus Θ is not injective. For the other implication take $\mathcal{X} \ni \chi_0 \neq \iota$ with $\chi_0 \in \ker \Theta$, i.e. $\Theta(\chi_0) = \text{int } \mathcal{A}$. Thus there exists a unitary $V \in \mathcal{U}(\mathcal{A})$ with $\text{ad } V = \text{ad } U_{\chi_0}$. From this we get $V^*U_{\chi_0} \in \mathcal{U}(\mathcal{A}^c) \cap \Pi_{\chi_0}(\mathcal{F})$, i.e. $\Pi_{\chi_0} \mathcal{A}^c \neq \emptyset$.

Finally, part (ii) follows from [7, Lemma 10.1.8]. ■

We mention several useful concepts for Hilbert C^* -systems $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ with a compact abelian group.

2.4 Definition $\beta \in \text{aut } \mathcal{A}$ is called a canonical automorphism if $\beta := \text{ad } V \upharpoonright \mathcal{A}, V \in \bigcup_{\chi \in \mathcal{X}} \mathcal{U}(\Pi_\chi \mathcal{F})$. The set of all canonical automorphisms is denoted by Γ .

2.5 Remark Note that for the set of canonical automorphisms we have $\text{int } \mathcal{A} \subseteq \Gamma \subseteq \text{aut } \mathcal{A}$ and that for $\alpha, \beta \in \Gamma$ the automorphisms $\alpha \circ \beta$ and $\beta \circ \alpha$ are unitarily equivalent. Furthermore, $\mathcal{X} \cong \Gamma / \text{int } \mathcal{A}$ and the set Γ is sometimes called *dual action* on \mathcal{A} .

For any $\gamma_1, \gamma_2 \in \Gamma$ we write

$$\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1} = \text{ad } \epsilon(\gamma_1, \gamma_2),$$

where $\epsilon(\gamma_1, \gamma_2) \in \mathcal{U}(\mathcal{A})$ and the class $\widehat{\epsilon}(\gamma_1, \gamma_2) := \epsilon(\gamma_1, \gamma_2) \text{ mod } \mathcal{U}(\mathcal{Z})$ is uniquely defined.

2.6 Lemma The permutators $\epsilon(\cdot, \cdot)$ satisfy the following relations:

$$\begin{aligned} \epsilon(\gamma_1, \gamma_2)\epsilon(\gamma_2, \gamma_1) &\equiv \mathbb{1} \text{ mod } \mathcal{U}(\mathcal{Z}), & \gamma_1, \gamma_2 \in \Gamma, \\ \epsilon(\iota, \gamma) &\equiv \epsilon(\gamma, \iota) \equiv \mathbb{1} \text{ mod } \mathcal{U}(\mathcal{Z}), & \gamma \in \Gamma, \\ \gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3) &\equiv \epsilon(\gamma_1\gamma_2, \gamma_3) \text{ mod } \mathcal{U}(\mathcal{Z}), & \gamma_1, \gamma_2, \gamma_3 \in \Gamma, \\ A\gamma_1(B)\epsilon(\gamma_1, \gamma_2) &\equiv \epsilon(\gamma'_1, \gamma'_2)B\gamma_2(A) \text{ mod } \mathcal{U}(\mathcal{Z}), & \gamma_1, \gamma_2, \gamma'_1, \gamma'_2 \in \Gamma \text{ and} \\ & & A \in (\gamma_1, \gamma'_1) \cap \mathcal{U}(\mathcal{A}), B \in (\gamma_2, \gamma'_2) \cap \mathcal{U}(\mathcal{A}). \end{aligned}$$

Proof: The first and second equations above are obvious. To prove the third one consider the inner automorphism characterized by the l.h.s. of the equation:

$$\begin{aligned} \text{ad}(\gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3)) &= \text{ad}(\gamma_1(\epsilon(\gamma_2, \gamma_3))) \circ \text{ad}(\epsilon(\gamma_1, \gamma_3)) \\ &= \gamma_1 \text{ad}(\epsilon(\gamma_2, \gamma_3)) \gamma_1^{-1} \circ \text{ad}(\epsilon(\gamma_1, \gamma_3)) \\ &= \gamma_1 (\gamma_2 \gamma_3 \gamma_2^{-1} \gamma_3^{-1}) \gamma_1^{-1} (\gamma_1 \gamma_3 \gamma_1^{-1} \gamma_3^{-1}) = (\gamma_1 \gamma_2) \gamma_3 (\gamma_1 \gamma_2)^{-1} \gamma_3^{-1} \\ &= \text{ad}(\epsilon(\gamma_1 \gamma_2, \gamma_3)), \end{aligned}$$

and this shows the desired relation. Finally, to prove the last equation recall that from the assumptions we have $\gamma'_1 = \text{ad}(A) \circ \gamma_1$ and $\gamma'_2 = \text{ad}(B) \circ \gamma_2$. From this we compute

$$\begin{aligned} \text{ad}(\epsilon(\gamma'_1, \gamma'_2)) &= (\text{ad}(A) \circ \gamma_1) \circ (\text{ad}(B) \circ \gamma_2) \circ (\text{ad}(A) \circ \gamma_1)^{-1} \circ (\text{ad}(B) \circ \gamma_2)^{-1} \\ &= \text{ad}(A) \circ \text{ad}(\gamma_1(B)) \circ \underbrace{\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1}}_{\text{ad}(\epsilon(\gamma_1, \gamma_2))} \circ \text{ad}(\gamma_2(A))^{-1} \circ \text{ad}(B)^{-1}. \end{aligned}$$

Therefore we get

$$\text{ad}(\epsilon(\gamma'_1, \gamma'_2) B \gamma_2(A)) = \text{ad}(A \gamma_1(B) \epsilon(\gamma_1, \gamma_2))$$

which implies the last equation of the statement. \blacksquare

2.7 Definition Let $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}$, with $\beta_\iota = \text{id}_\mathcal{A}$, be a system of representatives, i.e. $\pi(\beta_\chi) = \Theta(\chi)$. Then $\beta_\mathcal{X}$ is called a *lifting* of Θ if $\mathcal{X} \ni \chi \rightarrow \beta_\chi \in \text{aut } \mathcal{A}$ is a homomorphism.

2.8 Remark For the notion of lifting see for example Jones [16]. Sutherland [20, 21] says that Θ *splits* if there is a lifting of Θ . If Θ is an isomorphism then a lifting is also called *monomorphic section* (this latter name is used by Doplicher/Haag/Roberts [10]).

Results on the existence of liftings when \mathcal{A} is a von Neumann algebra and in a more general context w.r.t. the group \mathcal{X} (theory of Q-kernels) are due to Sutherland [20, 21]. Further, recall also the result of Doplicher/Haag/Roberts [10] in the “automorphism case” of the superselection theory, where $\mathcal{Z} = \mathbb{C} \mathbb{1}$ and \mathcal{A} is a so-called quasilocal algebra w.r.t. a net of local von Neumann algebras (see also [2]).

3 Hilbert extensions

The question concerning the description of $\{\mathcal{F}, \alpha_\mathcal{G}\}$ by \mathcal{A} and ‘something else’ is called the *reconstruction problem*. It is posed, for example, by Takesaki [23, p. 202] and by Bratteli/Robinson [8, p. 137]. Also the superselection structures in algebraic quantum field theory are connected with the reconstruction problem (for the automorphism case see Doplicher/Haag/Roberts [10]).

From Lemma 2.2 it seems natural to consider the corresponding *inverse problem*, which is an extension problem. This is just the emphasis in the mentioned papers by Sutherland and Jones (see also Nakamura/Takeda [19, 22]) as well as an essential aspect of the superselection theory (cf. [10, 2]).

3.1 Definition Let a system $\{\mathcal{A}, \Theta(\mathcal{X})\}$ be given where \mathcal{X} is a discrete abelian group and where $\Theta: \mathcal{X} \rightarrow \text{Out } \mathcal{A}$ is a homomorphism and put $\mathcal{G} := \hat{\mathcal{X}}$. A Hilbert C^* -system $\{\mathcal{F}, \alpha_\mathcal{G}\}$ is called a *Hilbert extension* of $\{\mathcal{A}, \Theta(\mathcal{X})\}$ if $\mathcal{A} = \Pi_\iota \mathcal{F}$ and $\Theta(\mathcal{X})$ coincides with the homomorphism given by Lemma 2.2.

Now let $\{\mathcal{A}, \Theta(\mathcal{X})\}$ and \mathcal{G} be given as in the previous definition. As it is pointed out, for example in [16], a crucial object for the extension problem is the so-called *obstruction* $\text{Ob } \Theta$. We recall the relevant relations: Choose a system $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}$, $\beta_\iota := \text{id}_\mathcal{A}$ of representatives. Then

$$\beta_{\chi_1} \circ \beta_{\chi_2} = \text{ad}(\omega(\chi_1, \chi_2)) \circ \beta_{\chi_1 \chi_2}, \quad (2)$$

where

$$\mathcal{X} \times \mathcal{X} \ni (\chi_1, \chi_2) \rightarrow \omega(\chi_1, \chi_2) \in \mathcal{U}(\mathcal{A}) \quad (3)$$

and we have the intertwining property

$$\omega(\chi_1, \chi_2) \in (\beta_{\chi_1 \chi_2}, \beta_{\chi_1} \circ \beta_{\chi_2}), \quad (4)$$

which is implied by (2). Moreover we have

$$\omega(\iota, \chi) = \omega(\chi, \iota) = \mathbb{1}. \quad (5)$$

Now associativity yields

$$\text{ad}(\omega(\chi_1, \chi_2)\omega(\chi_1 \chi_2, \chi_3)) = \text{ad}(\beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2 \chi_3))$$

so that there is $\gamma(\chi_1, \chi_2, \chi_3) \in \mathcal{U}(\mathcal{Z})$ with

$$\omega(\chi_1, \chi_2)\omega(\chi_1 \chi_2, \chi_3) = \gamma(\chi_1, \chi_2, \chi_3)\beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2 \chi_3).$$

If $\gamma(\chi_1, \chi_2, \chi_3) = \mathbb{1}$ for all $\chi_1, \chi_2, \chi_3 \in \mathcal{X}$ we obtain the equation

$$\omega(\chi_1, \chi_2)\omega(\chi_1 \chi_2, \chi_3) = \beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2 \chi_3). \quad (6)$$

Obviously, the existence of a system of representatives $\beta_{\mathcal{X}}$ such that equation (6) has a solution ω equipped with the properties (3)–(5) is necessary for the existence of a Hilbert extension. Even more, the existence of such a solution is also sufficient for the existence of a Hilbert extension.

3.2 Definition *A function ω , assigned to a given system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$, equipped with the properties (3)–(6) is called a generalized 2-cocycle.*

One calculates easily that the existence of a generalized 2-cocycle is independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives. Further, a generalized cocycle ω for $\beta_{\mathcal{X}}$ satisfies the relation

$$\text{ad}(\omega(\chi_1, \chi_2)\omega(\chi_2, \chi_1)^{-1}) = \beta_{\chi_1} \circ \beta_{\chi_2} \circ \beta_{\chi_1}^{-1} \circ \beta_{\chi_2}^{-1}.$$

The existence of a lifting of Θ can be expressed in terms of generalized 2-cocycles as follows.

3.3 Lemma *There exists a lifting $\beta_{\mathcal{X}}$ of Θ iff to each system $\gamma_{\mathcal{X}}$ of representatives there corresponds a generalized 2-cocycle ω of the form*

$$\omega(\chi_1, \chi_2) \equiv \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1 \chi_2} \pmod{\mathcal{U}(\mathcal{Z})},$$

where $V_{\chi} \in \mathcal{U}(\mathcal{A})$, $V_{\iota} = \mathbb{1}$. In this case, i.e. if there is a lifting $\beta_{\mathcal{X}}$, then a corresponding generalized 2-cocycle ω is given by $\omega(\chi_1, \chi_2) = \mathbb{1}$ for all $\chi_1, \chi_2 \in \mathcal{X}$.

Proof: Let $\beta_{\chi} = \text{ad}(V_{\chi}) \circ \gamma_{\chi}$, $V_{\chi} \in \mathcal{U}(\mathcal{A})$, $\chi \in \mathcal{X}$. Now if $\omega(\chi_1, \chi_2) = \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1 \chi_2}Z$ for some $Z \in \mathcal{U}(\mathcal{Z})$, then we have on the one hand $\beta_{\chi_1 \chi_2} = \text{ad}(V_{\chi_1 \chi_2}) \circ \gamma_{\chi_1 \chi_2}$ and on the other

$$\beta_{\chi_1} \circ \beta_{\chi_2} = (\text{ad}(V_{\chi_1}) \circ \gamma_{\chi_1}) \circ (\text{ad}(V_{\chi_2}) \circ \gamma_{\chi_2}) = \text{ad}(V_{\chi_1} \gamma_{\chi_1}(V_{\chi_2})\omega(\chi_1, \chi_2)) \circ \gamma_{\chi_1 \chi_2},$$

which using the assumption on ω and the fact that $\text{ad}(V_{\chi_1 \chi_2}Z) = \text{ad}(V_{\chi_1 \chi_2})$, implies that $\beta_{\chi_1 \chi_2} = \beta_{\chi_1} \circ \beta_{\chi_2}$, i.e. there is a lift of Θ . To prove the converse let $\beta_{\chi_1 \chi_2} = \beta_{\chi_1} \circ \beta_{\chi_2}$, so that from the above relations we have

$$\text{ad}(V_{\chi_1 \chi_2}) = \text{ad}(V_{\chi_1} \gamma_{\chi_1}(V_{\chi_2})\omega(\chi_1, \chi_2)),$$

which implies $\omega(\chi_1, \chi_2) = \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1 \chi_2} \pmod{\mathcal{U}(\mathcal{Z})}$. ■

3.4 Theorem *Let ω be a generalized 2-cocycle for the system $\beta_{\mathcal{X}}$ of representatives. Then there is a Hilbert extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Theta(\mathcal{X})\}$.*

Proof: The proof consists of several steps that correspond to gradually imposing a richer structure on an initially considered \mathcal{A} -left module:

1. Indeed, choose first system of 1-dimensional linear spaces, generated by abstract elements U_{χ} , $\chi \in \mathcal{X}$, $U_{\mathbb{1}} := \mathbb{1} \in \mathcal{A}$. Form the \mathcal{A} -left modules $\mathcal{A} \otimes \mathbb{C}U_{\chi}$ and $\mathcal{F}_0 := \bigoplus_{\chi} (\mathcal{A} \otimes \mathbb{C}U_{\chi})$. By identification $A \otimes \mathbb{1} \leftrightarrow A$, $\mathbb{1} \otimes U_{\chi} \leftrightarrow U_{\chi}$ one has

$$\mathcal{F}_0 = \left\{ \sum_{\chi, \text{ finite sum}} A_{\chi} U_{\chi} \mid A_{\chi} \in \mathcal{A} \right\},$$

where $\{U_{\chi} \mid \chi \in \mathcal{X}\}$ forms an abstract \mathcal{A} -module basis.

2. Next we want to equip \mathcal{F}_0 with a multiplication structure. First \mathcal{F}_0 becomes an \mathcal{A} -bimodule extending linearly the following definition

$$U_{\chi} A := \beta_{\chi}(A) U_{\chi}, \quad A \in \mathcal{A}, \chi \in \mathcal{X},$$

where $\beta_{\mathcal{X}}$ is the system of representatives to which we associate the generalized cocycle ω . Now the product structure is finally specified by putting

$$U_{\chi_1} \cdot U_{\chi_2} := \omega(\chi_1, \chi_2) U_{\chi_1 \chi_2}, \quad \chi_1, \chi_2 \in \mathcal{X},$$

where the cocycle equation (6) guarantees that the product is associative and the boundary conditions (5) lead to $U_{\chi} \cdot \mathbb{1} = \mathbb{1} \cdot U_{\chi} = U_{\chi}$. Note that the preceding product structure already implies that the U_{χ} are invertible. Indeed, it can be checked easily that the inverse is given explicitly by

$$U_{\chi}^{-1} := \beta_{\chi^{-1}}(\omega(\chi, \chi^{-1})^{-1}) U_{\chi^{-1}}$$

(use for example the relation $\beta_{\chi}(\omega(\chi^{-1}, \chi)) = \omega(\chi, \chi^{-1})$, which follows from the cocycle equation (6) by putting $\chi_1 := \chi$, $\chi_2 := \chi^{-1}$ and $\chi_3 = \chi$).

3. The following step consists in defining a $*$ -structure on \mathcal{F}_0 . This is done by putting

$$U_{\chi}^* := \omega(\chi^{-1}, \chi)^* U_{\chi^{-1}} \quad \text{and} \quad (AU_{\chi})^* := U_{\chi}^* A^*.$$

We still have to check that this definition is consistent, in particular with the product structure in \mathcal{F}_0 , i.e. we have to verify:

$$(U_{\chi}^*)^* = U_{\chi}, \quad (U_{\chi} A)^* = A^* U_{\chi}^* \quad \text{and} \quad (U_{\chi_1} \cdot U_{\chi_2})^* = U_{\chi_2}^* \cdot U_{\chi_1}^*. \quad (7)$$

For the first equation we have

$$\begin{aligned} (U_{\chi}^*)^* &= \left(\omega(\chi^{-1}, \chi)^* U_{\chi^{-1}} \right)^* = U_{\chi^{-1}}^* \omega(\chi^{-1}, \chi) = \omega(\chi, \chi^{-1})^* U_{\chi} \omega(\chi^{-1}, \chi) \\ &= \omega(\chi, \chi^{-1})^* \beta_{\chi}(\omega(\chi^{-1}, \chi)^*) U_{\chi} = \omega(\chi, \chi^{-1})^* \omega(\chi, \chi^{-1}) U_{\chi} \\ &= U_{\chi} \end{aligned}$$

The second equation in (7) can also be checked immediately from the definitions considered above. For the last equation we will consider the two sides separately: for the r.h.s. we have

$$\begin{aligned} U_{\chi_2}^* \cdot U_{\chi_1}^* &= \omega(\chi_2^{-1}, \chi_2)^* U_{\chi_2^{-1}} \cdot \omega(\chi_1^{-1}, \chi_1)^* U_{\chi_1^{-1}} \\ &= \omega(\chi_2^{-1}, \chi_2)^* \beta_{\chi_2^{-1}}(\omega(\chi_1^{-1}, \chi_1)^*) U_{\chi_2^{-1}} U_{\chi_1^{-1}} \\ &= \omega(\chi_2^{-1}, \chi_2)^* \beta_{\chi_2^{-1}}(\omega(\chi_1^{-1}, \chi_1)^*) \omega(\chi_2^{-1}, \chi_1^{-1}) U_{(\chi_1 \chi_2)^{-1}} \\ &= \omega(\chi_2^{-1}, \chi_2)^* \omega((\chi_1 \chi_2)^{-1}, \chi_1)^* \underbrace{\omega(\chi_2^{-1}, \chi_1^{-1})^* \omega(\chi_2^{-1}, \chi_1^{-1})}_{\mathbb{1}} U_{(\chi_1 \chi_2)^{-1}}, \end{aligned}$$

where we have used the relation

$$\beta_{\chi_2^{-1}}(\omega(\chi_1^{-1}, \chi_1)) = \omega(\chi_2^{-1}, \chi_1^{-1}) \omega(\chi_2^{-1} \chi_1^{-1}, \chi_1),$$

which again follows from the cocycle equation (6) taking now $\chi_1 := \chi_2^{-1}$, $\chi_2 := \chi_1^{-1}$ and $\chi_3 = \chi_1$. Now the l.h.s. reads

$$\begin{aligned} (U_{\chi_1} \cdot U_{\chi_2})^* &= U_{\chi_1 \chi_2}^* \omega(\chi_1, \chi_2)^* = \omega((\chi_1 \chi_2)^{-1}, \chi_1 \chi_2)^* U_{(\chi_1 \chi_2)^{-1}} \omega(\chi_1, \chi_2)^* \\ &= \omega((\chi_1 \chi_2)^{-1}, \chi_1 \chi_2)^* \beta_{(\chi_1 \chi_2)^{-1}}(\omega(\chi_1, \chi_2)^*) U_{(\chi_1 \chi_2)^{-1}}. \end{aligned}$$

Thus to show the last equation in (7) we need to prove that

$$\omega((\chi_1 \chi_2)^{-1}, \chi_1 \chi_2)^* \beta_{(\chi_1 \chi_2)^{-1}}(\omega(\chi_1, \chi_2)^*) = \omega(\chi_2^{-1}, \chi_2)^* \omega((\chi_1 \chi_2)^{-1}, \chi_1)^*$$

or taking adjoints

$$\beta_{(\chi_1 \chi_2)^{-1}}(\omega(\chi_1, \chi_2)) \omega((\chi_1 \chi_2)^{-1}, \chi_1 \chi_2) = \omega((\chi_1 \chi_2)^{-1}, \chi_1) \omega(\chi_2^{-1}, \chi_2).$$

But the preceding equation is nothing else than the cocycle equation (7) with $\chi_1 := (\chi_1 \chi_2)^{-1}$, $\chi_2 := \chi_1$ and $\chi_3 := \chi_2$. Finally, note that since $\beta_{\chi^{-1}}(\omega(\chi, \chi^{-1})^{-1}) = \omega(\chi^{-1}, \chi)^*$ we also have that the U_χ , are unitary, i.e. $U_\chi^* = U_\chi^{-1}$, $\chi \in \mathcal{X}$.

4. Here we will define a representation of the compact abelian group $\mathcal{G} = \widehat{\mathcal{X}}$ in terms of automorphisms of the $*$ -algebra \mathcal{F}_0 . The automorphisms are fixed by putting

$$\alpha_g(U_\chi) := \chi(g) U_\chi \quad \text{and} \quad \alpha_g(A U_\chi) := A \alpha_g(U_\chi) = \chi(g) A U_\chi, \quad g \in \mathcal{G}, A \in \mathcal{A}, \chi \in \mathcal{X}.$$

First we check that with the definition above the α_g is indeed an automorphism compatible with the structure in \mathcal{F}_0 :

$$\begin{aligned} \alpha_g(U_{\chi_1} U_{\chi_2}) &= \alpha_g(\omega(\chi_1, \chi_2) U_{\chi_1 \chi_2}) = (\chi_1 \chi_2)(g) \omega(\chi_1, \chi_2) U_{\chi_1 \chi_2} \\ &= \chi_1(g) \chi_2(g) U_{\chi_1} U_{\chi_2} = \alpha_g(U_{\chi_1}) \alpha_g(U_{\chi_2}) \end{aligned}$$

and

$$\begin{aligned} \alpha_g(U_\chi^*) &= \alpha_g(\omega(\chi^{-1}, \chi)^* U_{\chi^{-1}}) = (\chi^{-1})(g) \omega(\chi^{-1}, \chi)^* U_{\chi^{-1}} \\ &= \overline{\chi}(g) U_\chi^* = \alpha_g(U_\chi)^*. \end{aligned}$$

It can be also easily seen that the assignment $\mathcal{G} \ni g \rightarrow \alpha_g \in \text{aut } \mathcal{F}_0$ is an injective group homomorphism. Finally, note that the fixed point algebra of the previous action coincides with \mathcal{A} , i.e. for $F \in \mathcal{F}_0$, $\alpha_g(F) = F$ for all $g \in \mathcal{G}$ iff $F \in \mathcal{A}$. Indeed, for an arbitrary element $\sum_\chi A_\chi U_\chi \in \mathcal{F}_0$ the equation $\sum_\chi \chi(g) A_\chi U_\chi = \sum_\chi A_\chi U_\chi$, $g \in \mathcal{G}$, implies by the base property of the U_χ that $\chi(g) A_\chi = A_\chi$, $g \in \mathcal{G}$, $\chi \in \mathcal{X}$. Therefore if $\chi_0 \neq \iota$, then there is a $g_0 \in \mathcal{G}$ with $\chi_0(g_0) \neq 1$ and this shows that $A_{\chi_0} = 0$. The converse implication is obvious.

5. Finally, to specify a C^* -norm on \mathcal{F}_0 we introduce the following \mathcal{A} -valued scalar product (note the variation w.r.t. the definition in [2, p. 101]):

$$\langle F_1, F_2 \rangle := \sum_\chi \beta_\chi^{-1}(A_\chi^* B_\chi), \quad \text{where} \quad F_1 = \sum_\chi A_\chi U_\chi, \quad F_2 = \sum_\chi B_\chi U_\chi \in \mathcal{F}_0.$$

This scalar product satisfies the properties

$$\langle F_1, F_2 \rangle^* = \langle F_2, F_1 \rangle, \quad \langle F_1, F_1 \rangle \geq 0 \quad \text{and} \quad \langle F_1, F_1 \rangle = 0 \text{ iff } F_1 = 0.$$

Next we show that

$$\langle F_1, F_2 \rangle = \Pi_\iota(F_1^* F_2),$$

Indeed, using the definitions above we have

$$F_1^* F_2 = \sum_{\chi_1, \chi_2} U_{\chi_1}^* A_{\chi_1}^* B_{\chi_2} U_{\chi_2} = \sum_{\chi_1, \chi_2} \omega(\chi_1^{-1}, \chi_1)^* \beta_{\chi_1^{-1}}(A_{\chi_1}^* B_{\chi_2}) \omega(\chi_1^{-1}, \chi_2) U_{\chi_1^{-1} \chi_2}.$$

Putting, $\chi_1 = \chi_2 = \chi$ in the preceding expression we get

$$\begin{aligned} \Pi_\iota(F_1^* F_2) &= \sum_{\chi} \omega(\chi^{-1}, \chi)^* \beta_{\chi^{-1}}(A_{\chi}^* B_{\chi}) \omega(\chi^{-1}, \chi) \\ &= \sum_{\chi} \omega(\chi^{-1}, \chi)^* \omega(\chi^{-1}, \chi) \beta_{\chi}^{-1}(A_{\chi}^* B_{\chi}) \omega(\chi^{-1}, \chi)^* \omega(\chi^{-1}, \chi) \\ &= \langle F_1, F_2 \rangle, \end{aligned}$$

where for the second equation before we have used eq. (2) in the form $\beta_{\chi^{-1}} = \text{ad}(\omega(\chi^{-1}, \chi)) \circ \beta_{\chi}^{-1}$. In particular the relation above implies the following invariance property: $\langle \alpha_g(F_1), \alpha_g(F_2) \rangle = \langle F_1, F_2 \rangle$, $g \in \mathcal{G}$.

Define next the following norm on \mathcal{F}_0 by

$$|F| := \|\langle F, F \rangle\|^{\frac{1}{2}}, \quad F \in \mathcal{F}_0,$$

and the representation of \mathcal{F}_0 on $(\mathcal{F}_0, |\cdot|)$ in terms of multiplication operators

$$\rho(F)X := FX, \quad F, X \in \mathcal{F}_0.$$

Note that by the definition of the \mathcal{A} -valued scalar product the property $\rho(F^*) = \rho(F)^*$, $F \in \mathcal{F}_0$, holds. Now using the corresponding operator norm we introduce

$$\|F\|_* := |\rho(F)|_{op}, \quad F \in \mathcal{F}_0,$$

which by similar arguments as in [2, p. 102-103] satisfies the C^* -property $\|F^* F\|_* = \|F\|_*^2$. Further, it satisfies also (cf. again the previous reference)

$$\|A\|_* = \|A\|, \quad A \in \mathcal{A} \quad \text{and} \quad \|\alpha_g(F)\|_* = \|F\|_*, \quad g \in \mathcal{G}, F \in \mathcal{F}_0.$$

Therefore, we can finally extend α_g isometrically from \mathcal{F}_0 to

$$\mathcal{F} := \text{clo}_{\|\cdot\|_*}(\mathcal{F}_0).$$

Further, $\alpha_g \subset \text{aut } \mathcal{F}$ is norm continuous w.r.t. the pointwise norm convergence, because for any $F_0 = \sum_{\chi} A_{\chi} U_{\chi} \in \mathcal{F}_0$ we have

$$\|\alpha_{g_1}(F_0) - \alpha_{g_2}(F_0)\|_* = \left\| \sum_{\chi} (\chi(g_1) - \chi(g_2)) A_{\chi} U_{\chi} \right\|_* \leq \sum_{\chi} |\chi(g_1) - \chi(g_2)| \|A_{\chi}\|.$$

By construction we also have that $U_{\chi} \in \Pi_{\chi}(\mathcal{F})$, $\chi \in \mathcal{X}$. Therefore from the definitions of Sections 2 and 3 we have constructed a Hilbert C^* -extension $\{\mathcal{F}, \alpha_g\}$ of $\{\mathcal{A}, \Gamma\}$ and the proof is concluded. \blacksquare

Using now Lemma 3.3 one has

3.5 Corollary *If there is a lifting of Θ , then there is a Hilbert extension of $\{\mathcal{A}, \Theta(\mathcal{X})\}$, corresponding to $\omega = \mathbb{1}$.*

3.6 Remark The construction in the proof of the previous theorem generalizes to the nontrivial center situation the procedure already presented (with small modifications) in [2, Section 3.6].

The second problem consists in the description of all Hilbert extensions. For this purpose let $\Omega(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$ -valued 2-cocycles λ , i.e. λ satisfies equation (6) and condition (5), but (3),(4) are replaced by $\lambda(\chi_1, \chi_2) \in \mathcal{U}(\mathcal{Z})$. For example, $\lambda(\chi_1, \chi_2) := \mathbb{1}$ for all $\chi_1, \chi_2 \in \mathcal{X}$ is such a cocycle. Further let $\Omega_0(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$ -valued coboundaries ∂Z , i.e.

$$\partial Z(\chi_1, \chi_2) := \frac{Z(\chi_1)\beta_{\chi_1}(Z(\chi_2))}{Z(\chi_1\chi_2)},$$

where $Z(\cdot)$ is a $\mathcal{U}(\mathcal{Z})$ -valued 1-cycle, $Z(\iota) = \mathbb{1}$. Then ∂Z is a $\mathcal{U}(\mathcal{Z})$ -valued 2-cocycle, $\Omega \supseteq \Omega_0$. As usual, Ω and Ω_0 are abelian groups w.r.t. pointwise multiplication and the second cohomology is given by $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}}) := \Omega/\Omega_0$.

Next we need the concept of \mathcal{A} -module isomorphism of Hilbert extensions.

3.7 Definition Let $\{\mathcal{F}^1, \alpha_{\mathcal{G}}^1\}, \{\mathcal{F}^2, \alpha_{\mathcal{G}}^2\}$ be Hilbert extensions of $\{\mathcal{A}, \Theta(\mathcal{X})\}$. They are called \mathcal{A} -module isomorphic if there is an algebraic isomorphism $\Phi: \mathcal{F}^1 \rightarrow \mathcal{F}^2$, with $\Phi(A) = A$ for all $A \in \mathcal{A}$ and $\Phi \circ \alpha_g^1 = \alpha_g^2 \circ \Phi$ for all $g \in \mathcal{G}$.

3.8 Theorem Let ω_0 be a generalized 2-cocycle. Then:

- (i) Each $\mathcal{U}(\mathcal{Z})$ -valued 2-cocycle λ yields a Hilbert extension generated by the generalized 2-cocycle $\omega := \lambda \cdot \omega_0$ and each Hilbert extension is generated by some $\mathcal{U}(\mathcal{Z})$ -valued 2-cocycle λ via $\omega := \lambda \cdot \omega_0$.
- (ii) Two Hilbert extensions are \mathcal{A} -module isomorphic iff the generating generalized 2-cocycles ω_1, ω_2 differ only by a $\mathcal{U}(\mathcal{Z})$ -valued coboundary ∂Z , i.e. $\omega_1 = \partial Z \cdot \omega_2$.

Proof: (i) If two generalized 2-cocycles ω_1, ω_2 are given, then note first that $\lambda(\chi_1, \chi_2) := \omega_1(\chi_1, \chi_2)\omega_2(\chi_1, \chi_2)^{-1} \in \mathcal{U}(\mathcal{Z})$ for all χ_1, χ_2 , because of condition (4). Further, eq. (5) follows from the corresponding properties of ω_1 and ω_2 . Finally, the cocycle equation for $\lambda(\chi_1, \chi_2)$ is a consequence of the following computation:

$$\begin{aligned} \lambda(\chi_1, \chi_2)\lambda(\chi_1\chi_2, \chi_3) &= \omega_1(\chi_1, \chi_2)\omega_2(\chi_1, \chi_2)^{-1} \cdot \omega_1(\chi_1\chi_2, \chi_3)\omega_2(\chi_1\chi_2, \chi_3)^{-1} \\ &= \omega_1(\chi_1, \chi_2)\omega_1(\chi_1\chi_2, \chi_3)\omega_2(\chi_1\chi_2, \chi_3)^{-1}\omega_2(\chi_1, \chi_2)^{-1} \\ &= (\omega_1(\chi_1, \chi_2)\omega_1(\chi_1\chi_2, \chi_3)) \cdot (\omega_2(\chi_1, \chi_2)\omega_2(\chi_1\chi_2, \chi_3))^{-1} \\ &= \beta_{\chi_1}(\omega_1(\chi_2, \chi_3))\omega_1(\chi_1, \chi_2\chi_3) \cdot (\beta_{\chi_1}(\omega_2(\chi_2, \chi_3))\omega_2(\chi_1, \chi_2\chi_3))^{-1} \\ &= \beta_{\chi_1}(\omega_1(\chi_2, \chi_3))\omega_1(\chi_1, \chi_2\chi_3)\omega_2(\chi_1, \chi_2\chi_3)^{-1}\beta_{\chi_1}(\omega_2(\chi_2, \chi_3))^{-1} \\ &= \beta_{\chi_1}(\omega_1(\chi_2, \chi_3)\omega_2(\chi_2, \chi_3)^{-1})\omega_1(\chi_1, \chi_2\chi_3)\omega_2(\chi_1, \chi_2\chi_3)^{-1} \\ &= \beta_{\chi_1}(\lambda(\chi_2, \chi_3)) \cdot \lambda(\chi_1, \chi_2\chi_3), \end{aligned}$$

i.e. if one fixes a generalized 2-cocycle ω_0 , then $\omega := \lambda \cdot \omega_0$ runs through all generalized 2-cocycles ω if λ runs through all $\mathcal{U}(\mathcal{Z})$ -valued 2-cocycles in $\Omega(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$.

(ii) Let $\{\mathcal{F}^1, \alpha_{\mathcal{G}}^1\}$ and $\{\mathcal{F}^2, \alpha_{\mathcal{G}}^2\}$ be two Hilbert extensions of $\{\mathcal{A}, \Theta(\mathcal{X})\}$ and denote the corresponding set of abstract unitaries by $\{U_{\chi} \mid \chi \in \mathcal{X}\}$ resp. $\{V_{\chi} \mid \chi \in \mathcal{X}\}$.

Suppose first that there exists coboundary $\partial Z \in \Omega_0(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$, where $\beta_{\mathcal{X}}$ is system of representatives in Θ , such that the corresponding generalized cocycles ω_1 and ω_2 satisfy $\omega_1 = \partial Z \cdot \omega_2$. In this case we will show that the extensions are isomorphic. Indeed, define the isomorphism by

$$\Phi(AU_{\chi}) := A Z(\chi) V_{\chi}, \quad A \in \mathcal{A}, \chi \in \mathcal{X},$$

and extend it by linearity to the corresponding left \mathcal{A} -module. Now Φ is even a $*$ -homomorphism between the $*$ -algebras \mathcal{F}_0^1 and \mathcal{F}_0^2 that are defined in step 3 of the proof of Theorem 3.4. This follows from the following computations:

$$\begin{aligned}
\Phi(U_\chi A) &= \Phi(\beta_\chi(A)U_\chi) = Z(\chi) V_\chi A = \Phi(U_\chi)\Phi(A), \\
\Phi(U_\chi U_{\chi'}) &= \Phi(\omega_1(\chi, \chi')U_{\chi\chi'}) = \partial Z(\chi, \chi') \cdot \omega_2(\chi, \chi') Z(\chi\chi') V_{\chi\chi'} \\
&= \frac{Z(\chi)\beta_\chi(Z(\chi'))}{Z(\chi\chi')} \cdot Z(\chi\chi') V_\chi V_{\chi'} = Z(\chi) V_\chi Z(\chi') V_{\chi'} = \Phi(U_\chi)\Phi(U_{\chi'}), \\
\Phi(U_\chi^*) &= \Phi(\omega_1(\chi^{-1}, \chi)^* U_{\chi^{-1}}) = \partial Z(\chi^{-1}, \chi)^* \cdot \omega_2(\chi^{-1}, \chi)^* Z(\chi^{-1}) V_{\chi^{-1}} \\
&= Z(\chi^{-1})^* \beta_{\chi^{-1}}(Z(\chi))^* Z(\chi^{-1}) \omega_2(\chi^{-1}, \chi)^* V_{\chi^{-1}} = (Z(\chi) V_\chi)^* = \Phi(U_\chi)^*,
\end{aligned}$$

where $\chi, \chi' \in \mathcal{X}$, $A \in \mathcal{A}$. Note further that on \mathcal{F}_0^1 we already have $\Phi \circ \alpha_g^1 = \alpha_g^2 \circ \Phi$, $g \in \mathcal{G}$, since for any $\chi \in \mathcal{X}$ we have

$$\Phi \circ \alpha_g^1(AU_\chi) = \chi(g) A Z(\chi) V_\chi = \alpha_g^2(A Z(\chi) V_\chi) = \alpha_g^2 \circ \Phi(AU_\chi).$$

Recall that Φ is a bijection between \mathcal{F}_0^1 and \mathcal{F}_0^2 and we will finish this part of the proof if we can also show that Φ is even an isometry w.r.t the corresponding C^* -norms, because in this case we can isometrically extend Φ to the desired Hilbert extension isomorphism $\Phi: \mathcal{F}^1 \rightarrow \mathcal{F}^2$. Now denote by $\langle \cdot, \cdot \rangle_k$ the \mathcal{A} -valued scalar products on \mathcal{F}_0^k , $k = 1, 2$, given in step 5 of the proof of Theorem 3.4. For any $F = \sum_\chi A_\chi U_\chi \in \mathcal{F}_0^1$, so that $\Phi(F) = \sum_\chi A_\chi Z(\chi) V_\chi \in \mathcal{F}_0^2$, we have the following invariance

$$\langle \Phi(F), \Phi(F) \rangle_2 = \sum_\chi \beta_\chi^{-1} \left(Z(\chi)^* A_\chi^* A_\chi Z(\chi) \right) = \sum_\chi \beta_\chi^{-1} \left(A_\chi^* A_\chi \right) = \langle F, F \rangle_1.$$

From this and recalling the definition of the C^* -norm again in step 5 of the proof of Theorem 3.4 we immediately get the desired isometry property:

$$\|\Phi(F)\|_* = \sup_{\substack{X_2 \in \mathcal{F}_0^2 \\ |X_2| \leq 1}} |\Phi(F)X_2| = \sup_{\substack{X_1 \in \mathcal{F}_0^1 \\ |X_1| \leq 1}} |\Phi(F)X_1| = \|F\|_*.$$

To prove the converse implication assume that $\Phi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ specifies the isomorphism of the Hilbert extensions. Use the unitaries $\{U_\chi \mid \chi \in \mathcal{X}\}$ and $\{V_\chi \mid \chi \in \mathcal{X}\}$ in \mathcal{F}_1 resp. \mathcal{F}_2 to define the unitary

$$Z(\chi) := \Phi(U_\chi) V_\chi^*, \quad \chi \in \mathcal{X},$$

that satisfies $Z(\iota) = \mathbb{1}$. Even more $Z(\chi) \in \mathcal{U}(\mathcal{Z})$, since for any $A \in \mathcal{A}$ we have

$$A Z(\chi) = \Phi(AU_\chi) V_\chi^* = \Phi(U_\chi \beta_\chi^{-1}(A)) V_\chi^* = \Phi(U_\chi) (A^* V_\chi)^* = Z(\chi) A.$$

Finally, for $\chi, \chi' \in \mathcal{X}$ we have

$$\begin{aligned}
Z(\chi\chi') &= \Phi(\omega_1(\chi, \chi')^{-1} U_\chi U_{\chi'}) \cdot V_{\chi'}^* V_\chi^* (\omega_2(\chi, \chi')^{-1})^* \\
&= \omega_1(\chi, \chi')^{-1} \Phi(U_\chi) Z(\chi') V_\chi^* \omega_2(\chi, \chi') \\
&= \omega_1(\chi, \chi')^{-1} \Phi(U_\chi) \left(\beta_\chi(Z(\chi')^*) V_\chi \right)^* \omega_2(\chi, \chi') \\
&= \omega_1(\chi, \chi')^{-1} Z(\chi) \beta_\chi(Z(\chi')) \omega_2(\chi, \chi').
\end{aligned}$$

Now recalling the definition of the coboundary ∂Z , the preceding equations imply that $\omega_1(\chi, \chi') = \partial Z(\chi, \chi') \cdot \omega_2(\chi, \chi')$, $\chi, \chi' \in \mathcal{X}$, and the prove is concluded. \blacksquare

- 3.9 Remark** (i) Note that the results are independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$. Theorem 3.8 means that there is a bijection between $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$ and the set of all \mathcal{A} -module isomorphism classes of Hilbert extensions of $\{\mathcal{A}, \Theta(\mathcal{X})\}$ if there is one extension. In other words, the theorem gives an *outer* characterization of $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \beta_{\mathcal{X}})$ by the set of all \mathcal{A} -module isomorphism classes of Hilbert extensions.
- (ii) For a closer analysis of the second cohomology in the special cases where $\Gamma \cong \mathbb{Z}_N$ and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ see [1]. Consider also the abstract results in [18, Chapter 4].

4 The case of a trivial center

In this case we have $\mathcal{Z} = \mathbb{C}\mathbb{1}$, thus $\mathcal{U}(\mathcal{Z}) = \mathbb{T}\mathbb{1}$ and this implies that two automorphisms $\alpha, \beta \in \Gamma$ are either unitarily equivalent or otherwise disjoint. The following result is a special case of the famous Doplicher/Roberts theorem (see [13, 3]) in the present automorphism context.

4.1 Proposition *If there is a system of representatives $\epsilon(\alpha, \beta)$ of the permutator classes $\hat{\epsilon}(\alpha, \beta)$ which satisfy the equations*

$$\begin{aligned} \epsilon(\gamma_1, \gamma_2)\epsilon(\gamma_2, \gamma_1) &= \mathbb{1}, \\ \epsilon(\iota, \gamma) &= \epsilon(\gamma, \iota) = \mathbb{1}, \\ \gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3) &= \epsilon(\gamma_1\gamma_2, \gamma_3), \\ A\beta_{\chi_1}(B)\epsilon(\chi_1, \chi_2) &= \epsilon'(\chi_1, \chi_2)B\beta_{\chi_2}(A), \end{aligned}$$

for all $A \in (\beta_{\chi_1}, \beta'_{\chi_1})$, $B \in (\beta_{\chi_2}, \beta'_{\chi_2})$, where ϵ' belongs to $\beta'_{\mathcal{X}}$, then there is a generalized 2-cocycle ω_0 w.r.t. some system β_{χ} of representatives of the classes $\chi \in \Gamma/\text{int}\mathcal{A}$, with

$$\omega_0(\chi_1, \chi_2)\omega_0(\chi_2, \chi_1)^{-1} = \epsilon(\beta_{\chi_1}, \beta_{\chi_2}).$$

In this case there is a Hilbert extension \mathcal{F} of $\{\mathcal{A}, \Gamma\}$.

Conversely, if there is a Hilbert extension \mathcal{F} of $\{\mathcal{A}, \Gamma\}$, then to each $\alpha \in \Gamma$ there corresponds a unitary $V_{\alpha} \in \bigcup_{\chi \in \mathcal{X}} \mathcal{U}(\Pi_{\chi}\mathcal{F})$, such that $\alpha = \text{ad} V_{\alpha}|_{\mathcal{A}}$ and

$$\epsilon(\alpha, \beta) := V_{\alpha} V_{\beta} V_{\alpha}^{-1} V_{\beta}^{-1},$$

is a system of representatives of the permutators $\hat{\epsilon}(\alpha, \beta)$ satisfying the equations above.

4.2 Remark (i) In the present case the 2-cocycles λ of the preceding section are $\mathbb{T}\mathbb{1}$ -valued and the relation (6) becomes the usual cocycle equation

$$\lambda(\chi_1, \chi_2)\lambda(\chi_1\chi_2, \chi_3) = \lambda(\chi_2, \chi_3)\lambda(\chi_1, \chi_2\chi_3).$$

- (ii) In the particular case where \mathcal{A} is the inductive limit of a net of von Neumann algebras (which is a standard situation in algebraic quantum field theory, \mathcal{A} being the so-called quasilocal algebra) it can be shown that there is a lift $\gamma_{\mathcal{X}}$ of a given system of representatives $\beta_{\mathcal{X}}$, $\beta_{\chi} \in \chi$ (cf. Definition 2.7), and by Corollary 3.5 we have that $\omega(\chi_1, \chi_2) = 1$ is an admissible 2-cocycle of the system $\gamma_{\mathcal{X}}$. For a detailed construction of the lift see [10], [2, Section 3.2].

5 A Hilbert space representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$

Following Sutherland [20, 21] one can introduce a faithful Hilbert space representation of a Hilbert extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Theta(\mathcal{X})\}$.

First let \mathcal{H} be a Hilbert space and let π be a faithful representation of \mathcal{A} on \mathcal{H} . Form the Hilbert space $\mathcal{K} := l^2(\mathcal{X}, \mathcal{H})$ by completion of $C_0(\mathcal{X} \rightarrow \mathcal{H})$ w.r.t. the norm $\|f\|^2 := \sum_{\chi} \|f(\chi)\|_{\mathcal{H}}^2$. Choose a system $\beta(\mathcal{X})$ of representatives of $\Theta(\mathcal{X})$ and let ω be a corresponding generalized 2-cocycle such that $U_{\chi_1} \cdot U_{\chi_2} = \omega(\chi_1, \chi_2) U_{\chi_1 \chi_2}$. Now define a representation Φ of $\mathcal{F}_0 \subset \mathcal{F}$ on \mathcal{K} by

$$\begin{aligned} (\Phi(A)f)(\chi) &:= \pi(\beta_{\chi^{-1}}(A))f(\chi), \quad A \in \mathcal{A}, \\ \Phi(U_{\chi_0})f(\chi) &:= \pi(\omega(\chi^{-1}, \chi_0))f(\chi_0^{-1}\chi), \quad \chi_0 \in \mathcal{X}, \\ \Phi(AU_{\chi}) &:= \Phi(A)\Phi(U_{\chi}), \quad A \in \mathcal{A}, \chi \in \mathcal{X}. \end{aligned}$$

Note that $\Phi(\mathbb{1}) = \mathbb{1}_{\mathcal{K}}$ and $\|\Phi(A)\|_{\mathcal{K}} = \|A\|$. One calculates easily

$$\begin{aligned} \Phi(U_{\chi_1})\Phi(U_{\chi_2}) &= \Phi(\omega(\chi_1, \chi_2))\Phi(U_{\chi_1 \chi_2}), \\ \Phi(U_{\chi})\Phi(A) &= \Phi(\beta_{\chi}(A))\Phi(U_{\chi}), \\ \Phi(A^*) &= \Phi(A)^*, \quad \Phi(U_{\chi}^*) = \Phi(U_{\chi})^*. \end{aligned}$$

Further $\Phi(\sum_{\chi} A_{\chi} U_{\chi}) = 0$ implies $\sum_{\chi} A_{\chi} U_{\chi} = 0$, i.e. Φ is a *-isomorphism from \mathcal{F}_0 onto $\Phi(\mathcal{F}_0) \subset \mathcal{L}(\mathcal{K})$. Recall that

$$\|\Phi(F)\|_{\mathcal{K}} = \sup_{\|f\| \leq 1} \|\Phi(F)f\|_{\mathcal{K}}.$$

We have

5.1 Lemma *The relation*

$$\sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}} < \infty, \quad F \in \mathcal{F}_0, \quad (8)$$

holds.

Proof: With $F = \sum_{\chi} A_{\chi} U_{\chi}$ we have

$$\begin{aligned} \|\Phi(F)f\|^2 &= \sum_{y \in \mathcal{X}} \left\| \sum_{\chi} \pi(\alpha_{y^{-1}}(A_{\chi}) \omega(y^{-1}, \chi)) f(y^{-1}\chi) \right\|_{\mathcal{H}}^2 \\ &\leq \sum_{y \in \mathcal{X}} \left(\sum_{\chi} \left\| \pi(\alpha_{y^{-1}}(A_{\chi}) \omega(y^{-1}, \chi)) f(y^{-1}\chi) \right\| \right)^2 \\ &\leq \sum_{y \in \mathcal{X}} \left(\sum_{\chi} \|A_{\chi}\| \cdot \|f(y^{-1}\chi)\| \right)^2 \leq \sum_{y \in \mathcal{X}} \left(\sum_{\chi} \|A_{\chi}\|^2 \right) \left(\sum_{\chi} \|f(y^{-1}\chi)\|^2 \right) \\ &= \left(\sum_{\chi} \|A_{\chi}\|^2 \right) \sum_{\chi} \sum_{y \in \mathcal{X}} \|f(y^{-1}\chi)\|^2 = N(F) \|f\|^2 \sum_{\chi} \|A_{\chi}\|^2, \end{aligned}$$

where $N(F)$ denotes the number of terms of F . Hence we obtain

$$\|\Phi(F)\|_{\mathcal{K}} \leq N(F)^{1/2} \left(\sum_{\chi} \|A_{\chi}\|^2 \right)^{1/2} =: C_F.$$

and this implies

$$\|\Phi(\alpha_g F)\|_{\mathcal{K}} \leq C_F, \quad g \in \mathcal{G},$$

because the number of terms of $\alpha_g F$ equals that of F and $\|\chi(g)A_{\chi}\| = \|A_{\chi}\|$. This implies the inequality (8). \blacksquare

This result means that

$$\|\Phi(F)\|_{sup} := \sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}}$$

is a C*-norm on \mathcal{F}_0 .

5.2 Theorem *The relation*

$$\|\Phi(F)\|_{sup} = \|F\|_*, \quad F \in \mathcal{F}_0,$$

holds, and in particular $\|\Phi(F)\|_{\mathcal{K}} \leq \|F\|_*, F \in \mathcal{F}_0$.

Proof: The norm $\mathcal{F}_0 \ni F \rightarrow \|\Phi(F)\|_{sup}$ has the properties $\|\Phi(A)\|_{sup} = \|A\|$ for all $A \in \mathcal{A}$ and $\|\Phi(\alpha_g F)\|_{sup} = \|\Phi(F)\|_{sup}$ for all $g \in \mathcal{G}$. However, according to Doplicher/Roberts [11, p. 105] there is at most one C^* -norm on \mathcal{F}_0 with the mentioned properties. ■

5.3 Remark If there is a faithful state ϕ_0 of \mathcal{A} , then Theorem 5.2 can be improved. In this case

$$\|\Phi(F)\|_{\mathcal{K}} = \|F\|_*, \quad F \in \mathcal{F}_0,$$

holds. This is implied by the fact that in this case Sutherland's representation Φ of \mathcal{F}_0 on \mathcal{K} is unitarily equivalent to the so-called regular representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ (restricted to \mathcal{F}_0) given by the (faithful) GNS-representation π of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ on the GNS-Hilbert space \mathcal{H}_{π} w.r.t. the \mathcal{G} -invariant state $\phi(F) := \phi_0(\Pi_t F)$, $F \in \mathcal{F}$, such that $\|\Phi(F)\|_{\mathcal{K}} = \|\pi(F)\|_{\mathcal{H}_{\pi}}$ for all $F \in \mathcal{F}_0$, but $\|\pi(F)\|_{\mathcal{H}_{\pi}} = \|F\|_*$ for all $F \in \mathcal{F}$ (see, for example, [2, p. 108 ff.]).

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